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Multiple Markov Gaussian random fields

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Abstract

We are interested in ramdom fields $X(C)$ with parameter C, running through the class $\mathbf{C}= \{C; C \in C^2, \text{diffeomorphic to } S^1\}$. Referring to the canonical representation theory of Gaussian processes, developed by T. Hida, we generalize the theory to the case of our Gaussian random fields.

1 Introduction

We are interested in the way of dependency of random field $X(C)$ as a random complex system, where C runs through a class $\mathbf{C} = \{C; \text{diffeomorphic to } S^{d-1}, \text{convex}\}$.

In particular, we consider a Gaussian random field $X(C); C \in \mathbf{C}$, with a representation :

$$X(C) = \int_{(C)} F(C,u)x(u)du, \tag{1.1}$$

where x is an R^d-parameter white noise.

According to our purpose, we introduce a notion of Markov Gaussian random field which is a generalization of that of a Gaussian process $X(t)$ given by T. Hida (1960). Hida's definition for multiple Markov Gaussian processes can be treated even to the non-differentiable processes, different from the definition, given by J. L. Doob. Its generalization to $X(C)$ is possible because we can consider an increaing family of the C, so that a direction of evolution can be defined like in the case of linear parameter $t \in R^1$.

2 Preliminary

Consider Gaussian random fields

$$\{X(C); C \in \mathbf{C}\}$$

where

$$\mathbf{C} = \{C; C \in C^2, \text{diffeomorphic to } S^1, (C) \text{ is convex}\},$$
$$(C): \text{ being the domain enclosed by } C.$$

Assume that

1. $X(C) \neq 0$ for every C, and $E[X(C)] = 0$.
2. $\Gamma(C, C')$, for $C > C'$, admits variation in the variable C and that $\Gamma(C, C')$ never vanishes.

In particular, we consider the Gaussian random field $\{X(C); C \in \mathbf{C}\}$, with a representation

$$X(C) = \int_{(C)} F(C, u)x(u)du, \tag{2.1}$$

in terms of R^2−parameter white noise $x(u)$ and $L^2(R^2)$−kernel $F(C, u)$ for every C.

Definition (Canonical representation for a Gaussian random field)

Let $\mathrm{B}_{C'}(X)$ be the sigma field generated by $\{X(C), C < C'\}$. The representation (2.1) is called a canonical representation if

$$E[X(C)|\mathcal{B}_{C'}(X)] = \int_{(C')} F(C, u)x(u)du, \ C' < C. \tag{2.2}$$

Theorem 2.1 *The canonical representation is unique if it exists.*

Proof. See [7].

Definition(Martingale)

Let $\mathcal{B}_C(x) = \sigma\{< x, \xi >; \text{supp}\{\xi\} \subset (C)\}$. If

1. $E|X(C)| < \infty$ and
2. $E[X(C)|\mathcal{B}_{C'}(x)] = X(C')$, for any $C' < C$

then $X(C)$ is a martingale w.r.t. $\mathcal{B}_C(x)$.

Theorem 2.2 *If a Gaussian random field $Y(C)$, with mean zero, has a canonical representation and is a martingale, then there exists a locally square integrable function g such that*

$$Y(C) = \int_{(C)} g(u)x(u)du. \tag{2.3}$$

Proof. See [7].

Proposition *If $Y(C)$ is a martingale, never vanishes and $Y(C,x)$ is in the space (S), then $\mathcal{B}_C(x) = \mathcal{B}_C(Y)$.*

Definition (Markov property)

The Markov property for a random field $X(C)$ is defined by

$$P(X(C) \in B|\mathcal{B}_{C'}(X)) = P(X(C) \in B|X(C')). \tag{2.4}$$

Since $\{X(C)\}$ is Gaussian, it is sufficient to define the Markov property by

$$E(X(C)|\mathcal{B}_{C'}(X)) = E(X(C)|X(C')), \ C' \leq C. \tag{2.5}$$

Theorem 2.3 *Assume that $X(C)$ satisfies the Markov property then there exists $f \neq 0$ and $Y(C)$ which is a martingale w.r.t. $\mathcal{B}_C(Y)$ such that*

$$X(C) = f(C)Y(C). \tag{2.6}$$

Proof. See [7].

Corollary 1. *If, in addition, $Y(C,x) \in (S)$ then $Y(C)$ is a martingale w.r.t. $B_C(x)$.*

3 Multiple Markov property

Assume that the expression (1.1) for $X(C)$ is a canonical representation .

Definition For any choice of C_i's such that $C_0 \leq C_1 < \cdots < C_N < C_{N+1}$, if $E(X(C_i)|\mathcal{B}(C_0), i = 1, 2, \cdots, N$ are linearly independent and if $E(X(C_i)|\mathcal{B}(C_0), i = 1, 2, \cdots, N+1$ are linearly dependent then $X(C)$ is called N-ple Markov Gaussian random field.

Theorem 3.1 If $X(C)$ is $N-$ple Markov and it has a canonical representation, then it is of the form

$$X(C) = \int_{(C)} \sum_1^N f_i(C) g_i(u) x(u) du, \tag{3.1}$$

where the kernel $\sum f_i(C) g_i(u)$ is a Goursat kernel and $\{f_i(t)\}, i = 1, \cdots, N$ satisfies

$$\det(f_i(t_j)) \neq 0, for\ any\ N \text{different } t_j \tag{3.2}$$

and $\{g_i(u)\}, i = 1, \cdots, N$ are linearly independent in L^2-space.

Proof. Let

$$X(C) = \int_{(C)} F(C, u) x(u) du$$

be a canonical representation of $X(C)$ where $F(C, u)$ is a proper canonical kernel.

According to the assumption, $X(C)$ is an $N-$ple Markov process, we can prove that for any C_j with $C_1 < \cdots < C_N$, there exist coefficients $a_j(C; C_1, \cdots C_N)$ such that $[X(C) - \sum a_j(C; C_1, \cdots C_N) X(C_j)]$ is independent of $X(C'), C' \leq C_1$.

Thus we have

$$\int_{(C')} F(C', u) \left\{ F(C, u) - \sum_{j=1}^N a_j(C; C_1, \cdots, C_N) F(C_j, u) \right\} x(u) du.$$

Since $F(C', u)$ is a proper canonical kernel, we have

$$F(C, u) = \sum_{k=1}^N \mathbf{a}_k(C; C_1, \cdots, C_N) F(C_k, u) x(u) du. \tag{3.3}$$

Take N different $\{C'_j\}$ with $C'_1 < C'_2 \cdots < C'_N$, arbitrarily in the class $\mathbf{C}$. Using the expression of F as is in (3.2), we obtain

$$\begin{aligned} \sum_{j=1}^N a_j(C; C'_1, \cdots C'_N) F(C'_j, u) &= \sum_{j,k=1}^N a_k(C; C_1, \cdots, C_N) \\ &\quad a_j(C_k; C'_1, \cdots C'_N) F(C'_j, u). \end{aligned}$$

The $N-$ple Markov property of X implies the linearly independency of $\{F(C'_j, u), j = 1, \cdots, N\}$.

Thus we have

$$\sum_{j,k=1}^{N} a_k(C; C_1, \cdots, C_N) a_j(C_k; C'_1, C'_2, \cdots C'_N) = \sum_{j=1}^{N} a_j(C; C'_1, C'_2, \cdots C'_N), \tag{3.4}$$

for every j.

We can now prove that

$$\det(a_j(C_k; C'_1, C'_2, \cdots C'_N) \neq 0, \tag{3.5}$$

since $F(C_j, u) = \Sigma_{k=1}^{N} a_k(C_j; C'_1, C'_2, \cdots C'_N) F(C'_k, u),\ k = 1, \cdots, N$ are linearly independent functions. Then (3.3) becomes

$$\mathbf{a}(C, \underline{C}) = \mathbf{a}(C, \underline{C}') B(\underline{C}', \underline{C}) \tag{3.6}$$

where

$$\mathbf{a}(C, \underline{C}) = (a_j(C; C_1, C_2, \cdots, C_N);\ j = 1, \cdots, N)$$

and

$$B(\underline{C}, \underline{C}') = [b_{jk}(C_1, \cdots, C_N; C'_1, \cdots, C'_N),\ j, k = 1, \cdots, N],$$

with $\det(B(\underline{C}, \underline{C}')) \neq 0$.

For any $C'_j \in \mathbf{C},\ j = 1, \cdots, N$ such that $C'_j < C_j$,

$$\mathbf{a}(C, \underline{C}) = \mathbf{a}(C, \underline{C}') B(\underline{C}', \underline{C}) = \mathbf{a}(C, \underline{C}'') B(\underline{C}'', \underline{C}') B(\underline{C}', \underline{C}),$$

$$\mathbf{a}(C, \underline{C}') = \mathbf{a}(C, \underline{C}') B(\underline{C}', \underline{C}).$$

Hence

$$B(\underline{C}'', \underline{C}') B(\underline{C}', \underline{C}) = B(\underline{C}'', \underline{C}). \tag{3.7}$$

Let us take fixed C_j's and define $\mathbf{f}_{\underline{C}'}(C), \underline{C}' = (C'_1, C'_2, \cdots, C'_N)$,by

$$\mathbf{f}_{\underline{C}'}(C) = \mathbf{a}(C, \underline{C}') B(\underline{C}', \underline{C}),\ for\ C > C'_N.$$

where $\underline{C}'$ is an $N-$ple $(C'_1, \cdots, C'_N)$ such that $C_N > C_{N-1} > \cdots > C_1 > C'_N > C'_{N-1} > \cdots > C'_1$, thus we can see that $\mathbf{f}_{\underline{C}''}$ is an extension of $\mathbf{f}_{\underline{C}'}(C)$ if $C'_N > C'_{N-1} > \cdots, C'_1 > C''_N > C''_{N-1}, \cdots > C''_1$. It shows that there exists a common extension $\mathbf{f}(C) = (f_1(C), \cdots, f_N(C))$ for all $\mathbf{f}_{\underline{C}'}(C)$'s. Denote it by

$$\mathbf{f}(C) = (f_1(C), \cdots, f_N(C)).$$

We can see from (3.4) and the definition of $\mathbf{f}_{\underline{C}'}(C)$ that $f_i(C)$ satisfies (2.2).

Let us take a fixed curve $C_0 \in \mathbf{C}$. If $C > C_N > \cdots > C_1 > C'_N > \cdots > C'_1 > C_0$ then

$$\begin{aligned} F(C,u) &= \sum_{j=1}^{N} a_j(C; C_1, \cdots, C_N) F(C_j, u) \\ &= \mathbf{a}(C, \underline{C}) \mathbf{F}(\underline{C}, u)^* \\ &= \mathbf{f}(C) B(\underline{C}', \underline{C})^{-1} \mathbf{F}(\underline{C}, u)^* \\ &= \mathbf{f}(C) \mathbf{g}(u, \underline{C}', \underline{C})^*, \end{aligned}$$

where

$$\mathbf{F}(\underline{C}, u) = (F(C_1, u), \cdots, F(C_N, u))$$

and

$$\mathbf{g}(u, \underline{C}', \underline{C}) = \mathbf{F}(\underline{C}, u) B(\underline{C}', \underline{C})^{*-1}.$$

For $C > C''_N > \cdots > C''_1 > C'''_N > \cdots > C'''_1$, this is equal to

$$\mathbf{f}(C)\mathbf{g}(u, \underline{C}''', \underline{C}'')^*,$$

so that

$$\mathbf{f}(C)\mathbf{g}(u, \underline{C}', \underline{C})^* = \mathbf{f}(C)\mathbf{g}(u, \underline{C}''', \underline{C}'')^*,$$

for $C > C''_N, C_N$. Since $\mathbf{f}$ satisfies (2.2), we have

$$\mathbf{g}(u, \underline{C}', \underline{C})^* = \mathbf{g}(u, \underline{C}''', \underline{C}'')^*.$$

Thus $g(u) = \mathbf{g}(u, \underline{C}', \underline{C})$ is well defined as a function of u, and

$$F(C,u) = \mathbf{f}(C)\mathbf{g}(u)^* = \sum_{i=1}^{N} f_i(C) g_i(u),$$

where $\{g_i(u), i = 1, \cdots, N\}$ are linearly independent since $\{F(C_j, u)\}; j = 1, \cdots, N$, are linearly independent.

Corollary 3.1 If $X(C)$ is a $N-$ple Markov Gaussian random field, then the covariance function $\Gamma(C, C') = E[X(C)X(C')]$ can be expressed in the form

$$\sum_{i,j=1}^{N} f_i(C) f_j(C') h_{ij}(C, C'), \tag{3.8}$$

where the matrix $(h_{ij}(C, C'))$ is a Gramian and $h_{ij}(C, C')$ is a function of $(C) \cap (C')$.

Remark Gramian is a matrix $[(g_i, g_j)]$ where (g_i, g_j) is the inner product of g_i and g_j in L_2−space.

Corollary 3.2 If $N = 1$, then it is (simple) Markov.

Proof. It can be easily seen from the expression of canonical representation.

4 Application

Let $X(C)$ be a simple Markov Gaussian random field. The covariance function is of the form

$$\Gamma(C, C') = f(C)f(C')h(CC').$$

For $Y(C) = F(X(C)$ the variation of $Y(C)$ is obtained as

$$\begin{aligned} \delta Y(C)) &= \left(\Sigma \int_C \frac{\partial a_k}{\partial n}(s)\delta n(s)ds\right) H_k(Y(C), \sigma^2) \\ &+\Sigma a_k(C) H_{k-1}(Y(C), \sigma^2) \int_C f(s)x(s)\delta n(s)ds \end{aligned}$$

if

$$\Sigma\{\left(\int_C \frac{\partial a_k}{\partial n}(s)\delta n(s)ds\right)^2 + a_{k+1}^2\}\frac{\sigma^{2k}}{k!}$$

converges.

which is an analogue of Ito's formula. (see [7] for detail)

5 Concluding Remarks

1. Our definition is applied only for Gaussian case. A generalization may be possible for Poisson case, but we need some additional assumptions.

2. In principle, Markov property should be defined only by the way of dependency according as the parameter deforms, but not on the

analytic property of $X(C)$ in C, like as in the case of $X(t)$. Namely, it depends only on the observed values of the past. In this case, they are given by conditional expectation. For a field in question the conditional expectation depends on the observed values as many as N.

From the view point of prediction theory such a finite dependency property is significant.

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